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Density of $1/(1+x)$-Polynomials in $C[\gamma, \infty]$

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Let $\gamma \geqslant 0$, let $C[\gamma, \infty]$ be the space of continuous functions on $[\gamma, \infty]$, and let $\phi(x) = 1/(1+x)$. Then a ϕ-polynomial on $[\gamma, \infty]$ is an expression of the form

$$\sum_{k=1}^{n} a_k \phi(\alpha_k x) = \sum_{k=1}^{n} a_k/(1 + \alpha_k x), \qquad 1 + \alpha_k x > 0.$$

Sets of ϕ-polynomials are among the best known curves of functions [4, p. 91]. Best Chebyshev approximation by ϕ-polynomials on $[\gamma, \infty]$ is considered in [3]. Density results for ϕ-polynomials (on a finite range) follow from the theory of [2].

DEFINITION. A sequence of functions is *fundamental* in a subset G of $C[\alpha, \beta]$ if the set of (finite) linear combinations of the functions is dense in G (with respect to the Chebyshev norm on $[\alpha, \beta]$).

LEMMA. *Let $\gamma > 0$. Let $\{\beta_k\}$ be a positive sequence with limit zero and $\psi(x) = 1/(1 + 1/x) = x/(1+x) = 1 - 1/(1-x)$. The sequence of functions* $\{1, \psi(\beta_1 x), \psi(\beta_2 x),...\}$ is fundamental in $C[0, 1/\gamma]$. *The sequence of functions $\{\psi(\beta_1 x), \psi(\beta_2 x),...\}$ is fundamental in the continuous functions on* $[0, 1/\gamma]$ *which vanish at* 0.

Proof. ψ has a Taylor series about zero with all but the zeroth coefficient nonzero. Apply the theory of [2].

THEOREM. *Let $\gamma > 0$. Let $\{\alpha_k\}$ be a sequence with limit ∞. The set $\{1, \phi(\alpha_1 x), \phi(\alpha_2 x),...\}$ is fundamental in $C[\gamma, \infty]$.*

Proof. Let $f \in C[\gamma, \infty]$ and $\epsilon > 0$ be given. Let $y(x) = 1/x$ and define

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$g(y) = f(x) = f(1/y)$ for $0 \leqslant y < 1/\gamma$. g is in $C[0, 1/\gamma]$. Let $\beta_i = 1/\alpha_i$. By the above lemma, we have for some n, $a_1, \ldots, a_n$,

$$\left| g(y) - a_1 - \sum_{k=2}^{n} a_k/(1 + 1/(\beta_k y)) \right| < \epsilon \qquad 0 \leqslant y \leqslant 1/\gamma.$$

Since $1/(\beta_k y) = \alpha_k x$, we have

$$\left| f(x) - a_1 - \sum_{k=2}^{n} a_k/(1 + \alpha_k x) \right| < \epsilon \qquad \gamma \leqslant x \leqslant \infty,$$

proving the theorem.

By similar arguments we have

THEOREM. *Let $\gamma > 0$. Let $\{\alpha_k\}$ be a sequence with limit ∞. The set $\{\phi(\alpha_1 x), \phi(\alpha_2 x), \ldots\}$ is fundamental in the continuous functions on $[\gamma, \infty]$ vanishing at ∞.*

THEOREM. *Let $\{\delta_k\}$ be an increasing sequence with limit one. The set $\{1, \phi(\delta_1 x), \phi(\delta_2 x), \ldots\}$ is fundamental in $C[0, \infty]$.*

Proof. Let $\{\alpha_k\} \to \infty$ and $\beta_k = 1/\alpha_k$. Let $f \in C[0, \infty]$ and $\epsilon > 0$ be given. Let $y(x) = 1/(1 + x)$ for $0 \leqslant x \leqslant \infty$, which implies $x = (1/y) - 1$ for $0 \leqslant y \leqslant 1$. Define $g(y) = f(x) = f((1/y) - 1)$ for $0 \leqslant y \leqslant 1$. g is in $C[0, 1]$. By the previous lemma, there is n, $a_1, \ldots, a_n$ such that

$$\left| g(y) - a_1 - \sum_{k=2}^{n} a_k/(1 + 1/(\beta_k y)) \right| < \epsilon \qquad 0 \leqslant y \leqslant 1.$$

Since $1/(\beta_k y) = \alpha_k(1 + x)$, we have

$$\left| f(x) - a_1 - \sum_{k=2}^{n} a_k/(1 + \alpha_k(1 + x)) \right| < \epsilon, \qquad 0 \leqslant x \leqslant \infty,$$

that is,

$$\left| f(x) - a_1 - \sum_{k=2}^{n} [a_k/(1 + \alpha_k)]/[1 + (\alpha_k/(1 + \alpha_k))\, x] \right| < \epsilon, \qquad 0 \leqslant x \leqslant \infty.$$

Given δ_k in $(0, 1)$, we can choose α_k such that $\alpha_k/(1 + \alpha_k) = \delta_k$. The theorem is proven. By similar arguments we get

THEOREM. *Let $\{\delta_k\}$ be an increasing sequence with limit one. The set $\{\phi(\delta_1 x), \phi(\delta_2 x), \ldots\}$ is fundamental in the set of elements of $C[0, \infty]$ vanishing at ∞.*

Remark. Since the space of approximation is $[0, \infty]$ and we are approximating by ϕ-polynomials, consideration of a multiplicative change of variable shows that the two previous theorems hold for $\{\delta_k\}$ an increasing sequence with any positive limit.

A ϕ-polynomial can also be expressed in the form

$$\sum_{k=1}^{n} b_k/(\beta_k + x), \qquad \beta_k + x > 0.$$

Density results for these (on a finite range) are given by Achieser [1, 246 ff., 254 ff.]

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